353. Proposed by CLIFFORD N. MILLS, Brookings, North Dakota.

A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the center. Find the greatest bending moment and the greatest stress in the fibers. Take the specific gravity of oak as .934.

NUMBER THEORY.

270. Proposed by GERSHOM N. CARMICHAEL, Urbana, Ill.

Does there exist a fraction p/q in its lowest terms such that the ratio of the sum of the divisors of p to the sum of the divisors of q is equal to p/q? Give a method of finding such fractions not in their lowest terms.

271. Proposed by HORACE OLSON, Chicago, Ill.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

474. Proposed by A. A. BENNETT, University of Texas.

Show that the value of the infinite continued fraction, all of whose coefficients are unity,

$$1 + \frac{1}{1+}\,\frac{1}{1+}\cdots, \quad \text{is} \quad \tfrac{1}{2}(1 + \sqrt{5}).$$

Also find an explicit algebraic formula for the nth convergent.

SOLUTION BY C. C. YEN, Tangshan, North China.

1. Let x denote the value of the continued fraction. Then,

$$x = 1 + 1/x, \quad \text{i. e.,} \quad x^2 - x - 1 = 0,$$

therefore,

$$x = \tfrac{1}{2}(1 + \sqrt{5}),$$

where the sign for the radical is positive, since x is evidently positive.

2. Let p_n/q_n denote the nth convergent. Then

$$p_1 = 1, \qquad q_1 = 1, \qquad p_2 = 2, \qquad q_2 = 1. \tag{I}$$

Also, since all the coefficients are unity,

$$p_n = p_{n-1} + p_{n-2}, \qquad q_n = q_{n-1} + q_{n-2} \qquad (n = 3, 4, 5, \cdots). \tag{II}$$

And, it follows, therefore, from I and II,

$$q_n = p_{n-1} \qquad (n = 2, 3, 4, \cdots). \tag{III}$$

Now consider the series

$$1 + x + 2x^2 + p_3x^3 + p_4x^4 + \cdots + p_nx^n + \cdots,$$

where the coefficients p_n satisfy II. It is a recurring series whose scale of relation is $1 - x - x^2$, and whose generating function is $1/(1 - x - x^2)$. Hence, p_n is the coefficient of x^n of the expansion of this function; and, by III, q_n is the coefficient of x^{n-1} of the same expansion.

If we put $\alpha = -\tfrac{1}{2}(1 + \sqrt{5})$, $\beta = -\tfrac{1}{2}(1 - \sqrt{5})$, we get

$$\frac{1}{1 - x - x^2} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha - x} - \frac{1}{\beta - x}\right),$$

whence the general term of the expansion of $1/(1 - x - x^2)$ is

$$\frac{1}{\alpha - \beta}\left\{\left(\frac{1}{\alpha}\right)^{n+1} - \left(\frac{1}{\beta}\right)^{n+1}\right\}x^n = \frac{1}{\alpha - \beta}\left\{\frac{\beta^{n+1} - \alpha^{n+1}}{(\alpha\beta)^{n+1}}\right\}x^n.$$

But $\alpha\beta = -1$, and $q_n = p_{n-1}$. It follows, therefore, that

$$p_n = (-)^n \left\{ \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right\}, \qquad q_n = (-1)^{n-1} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}.$$

Hence, finally, the nth convergent is given by

$$-(\alpha^{n+1} - \beta^{n+1})/(\alpha^n - \beta^n),$$

where α, β are the roots of the equation $x^2 - x - 1 = 0$.

Also solved by PAUL CAPRON, N. P. PANDYA, and O. S. ADAMS.

475. Proposed by E. B. ESCOTT, Kansas City, Mo.

A man makes a contract to purchase a house, making a cash payment down and agreeing to make monthly payments of a dollars, interest being charged at 6 per cent., the balance of the monthly payments being credited on the principal. Find a formula for M_n, the balance due after n payments.

SOLUTION BY C. R. DUNCAN, Amherst, Massachusetts.

Let M_0 = balance due after the original cash payment, and r = rate of interest per month (= .06/12), then M_0r = interest due at end of first month, and $a - M_0r$ = amount paid back on the principal M_0.

At the end of the second month the interest would be less, the difference being equal to the interest on the amount paid back the first month, or $(a - M_0r)r$. But as all monthly payments are to be equal this amount would be credited on the principal. Hence, the amount paid back on the principal at the end of the second month is

$$(a - M_0r) + (a - M_0r)r = (a - M_0r)(1 + r).$$

Similarly, the amount paid back at the end of the third month would be equal to the amount paid back the second month plus the difference in interest between the second and third months, or

$$(a - M_0r)(1 + r) + (a - M_0r)(1 + r)r = (a - M_0r)(1 + r)^2.$$

Hence, the amount paid back at the end of the nth month $= (a - M_0r)(1 + r)^{n-1}$. Therefore, the total amount paid back in n months is

$$(a - M_0r) + (a - M_0r)(1 + r) + (a - M_0r)(1 + r)^2 + \cdots + (a - M_0r)(1 + r)^{n-1} = \frac{(a - M_0r)[(1 + r)^n - 1]}{r},$$

and the balance due after n payments is

$$M_n = M_0 - \frac{(a - M_0r)[(1 + r)^n - 1]}{r}.$$

Putting the right-hand member equal to 0 and solving for a, we have a formula for finding the amount of the monthly payments required to pay back the principal M_0 in a given number of months,

$$a = \frac{M_0r(1 + r)^n}{(1 + r)^n - 1}.$$

Also solved by G. W. HARTWELL, E. J. OGLESBY, HORACE OLSON, A. R. NAUER, PAUL CAPRON, G. PAASWELL, H. N. CARLETON, J. B. REYNOLDS, and the PROPOSER.

476. Proposed by W. HAROLD WILSON, University of Illinois.

Prove that if $x_h \neq x_j$ $\qquad (h, j = 1, 2, \cdots, n, h \neq j),$

then

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{h=1}^{n}{}' (x_i - x_h)} = 1,$$

where the prime indicates the omission of zero factors in the denominator.